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# Note

# The Use of the Odd–Even Hopscotch Algorithm for a Thermal Shock Problem

## 1. INTRODUCTION

The fast partial differential equation solver proposed by Gourlay [1] is a very attractive computational scheme. The implemented process is explicit, in the sense that systems of equations need not be solved at each stage, and unconditionally stable. Gourlay and McGuire [2] refer to the basic technique as "odd–even hopscotch." This note is concerned with the behaviour of the odd–even hopscotch solution for a thermal shock problem.

It is a common practive in the numerical solution of heat conduction problems to ignore discontinuities in the initial data. The error introduced is usually acceptable and often decays as the numerical scheme progresses. The behaviour of standard finite-differences techniques for problems containing discontinuities is fairly well understood (see [3]) and a similar appreciation has been made of finite-element methods (see [4]). However, the authors have observed some unusual behaviour in the odd-even hopscotch solution of the heat conduction equation with discontinuous initial data. The form of the solution obtained is dependent upon the implementation of the algorithm. When the initial approximation at the point adjacent to the discontinuity is implicit the subsequent solution appears smooth for all parameter values. However, if the initial approximation at that point is explicit a "wave-like" disturbance can propagate through the solution domain, resulting in very poor accuracy. For the parameter range in which both implementations produce "sensible" results a significant discrepancy may still be observed.

It is the purpose of this note to demonstrate this unusual behaviour for the benefit of those employing hopscotch techniques in the numerical solution of thermal problems.

## 2. PROBLEM SPECIFICATION AND METHOD OF SOLUTION

Consider the simple, non-dimensionalised, heat flow problem,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0; \tag{1}$$

$$u(x,0) = 0, \qquad 0 < x \leqslant 1; \tag{2}$$

$$u(1, t) = 0, \qquad t \geqslant 0; \tag{3}$$



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and

$$u(0, t) = 1, \qquad t \geqslant 0, \tag{4}$$

where $u$ represents temperature. The initial temperature distribution contains a jump discontinuity at $x = 0$. The equations typify the heating of a cooled, lagged rod by the rapid introduction of one end to a hot environment. The analytic solution of the problem defined by (1) to (4) is given by Carslaw and Jaeger [5].

The odd–even hopscotch approximation of Eq. (1) is

$$(1 - r\theta_q^{p+1}\delta_x^2)\, u_q^{p+1} = (1 + r\theta_q^p\delta_x^2)\, u_q^p, \qquad q = 1,..., 2n - 1, p = 0, 1,..., \tag{5}$$

where

$$\begin{aligned} \theta_q^p &= 1, \qquad p + q \text{ odd}, \\ &= 0, \qquad p + q \text{ even}, \end{aligned}$$

$u_q^p \simeq u(qh, pk)$, $h$ and $k$ are the parameters of the finite-difference mesh, $r = k/h^2$, $2nh = 1$ and $\delta_x$ is the central difference operator for the space variable. The initial and boundary conditions become

$$\begin{aligned} u_0^p &= 1, \qquad p = 0, 1,..., \\ u_{2n}^p &= 0, \qquad p = 0, 1,..., \\ u_q^0 &= 0, \qquad q = 1,..., 2n - 1. \end{aligned} \tag{6}$$

In order to investigate the behaviour of the solution of the finite-difference equations, the odd–even hopscotch algorithm is considered as a two-stage, explicit process where

$$\begin{aligned} (1 + 2r)\, u_q^{2m+2} = {} & (1 - 2r + 4r^2)\, u_q^{2m} + 2r(1 - 2r)(u_{q+1}^{2m} + u_{q-1}^{2m}) \\ & + 2r^2(u_{q+2}^{2m} + u_{q-2}^{2m}), \qquad m = 0, 1,..., \end{aligned} \tag{7}$$

for even points $q = 2i$, $i = 1, 2,..., n - 1$, and

$$\begin{aligned} (1 + 2r)^2\, u_q^{2m+2} = {} & (1 - 8r^3)\, u_q^{2m} + 2r(1 + 3r^2)(u_{q-1}^{2m} + u_{q+1}^{2m}) \\ & + 2r^2(1 - 2r)(u_{q-2}^{2m} + u_{q+2}^{2m}) \\ & + 2r^3(u_{q-3}^{2m} + u_{q+3}^{2m}), \qquad m = 0, 1,..., \end{aligned} \tag{8}$$

for internal, odd points $q = 2i + 1$, $i = 1, 2,..., n - 2$. Of the two remaining points only the solution at $q = 1$ is of direct interest as the disturbance under investigation emanates from the initial discontinuity. The equation corresponding to Eq. (8) for the point $q = 1$ is

$$\begin{aligned} (1 + 2r)^2\, u_1^{2m+2} = {} & (1 - 2r^2 - 4r^3)\, u_1^{2m} + 2r(1 + 2r^2)\, u_2^{2m} + 2r^2(1 - 2r)\, u_3^{2m} \\ & + 2r^3 u_4^{2m} + 2r(1 + r)^2, \qquad m = 0, 1,... . \end{aligned} \tag{9}$$

The equation at $q = 2n - 1$ has a similar form.

An examination of the solution to these equations, subject to the initial and boundary conditions (6), after just two time steps ($m = 0$) illustrates the effect of the discontinuity. Equations (6), (7), (8) and (9) yield

$$\begin{aligned} u_1^2 &= 2r(1 + r)^2/(1 + 2r)^2, \\ u_2^2 &= 2r^2/(1 + 2r), \\ u_3^2 &= 2r^3/(1 + 2r)^2, \end{aligned} \tag{10}$$

as the only non-zero values at time $t = 2k$. The propagation of non-zero values is illustrated in Fig. 1. For a value of $r$ of 1.4 or more two of the predicted temperatures are in excess of the boundary value. When the value of $r$ is greater than about 1.6, $u_3^2 < u_2^2 > u_1^2 > 1$ and a "wave-like" temperature profile is produced. The extent to which this spurious wave penetrates the solution domain depends on the size of $r$ but even for modest values of $r$ the effect upon the accuracy at later times is significant.

In the following section the behaviour of the leading non-zero values is examined in detail.

## 3. Difference Solution

Consider the behaviour of the leading non-zero values, $u_{2m-1}^{2m}$, $u_{2m}^{2m}$ and $u_{2m+1}^{2m}$, at even numbers of time steps. As illustrated in Fig. 1, many of the terms in (7) and (8) are zero for the points $q = 2m$ and $q = 2m + 1$ and it is easily shown from these two equations that

$$u_{2m+1}^{2m} = \frac{r}{(1 + 2r)} u_{2m}^{2m}, \qquad m = 1, \ldots, n - 1, \tag{11}$$

and hence that

$$u_{2m+2}^{2m+2} = 4\left(\frac{r}{1 + 2r}\right)^2 u_{2m}^{2m}, \qquad m = 1, \ldots, n - 2. \tag{12}$$

Using the appropriate expression given in Eq. (10) as an initial condition the above recurrence relation has the solution

$$u_{2m}^{2m} = r\left[\frac{2r}{1 + 2r}\right]^{2m-1}, \qquad m = 1, \ldots, n - 1.$$

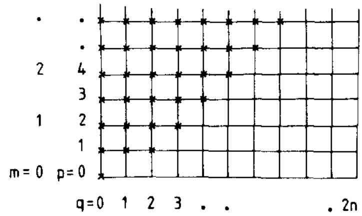


FIG. 1. Propagation of non-zero values ($x$ denotes non-zero value).

Clearly, the value $u_{2m}^{2m}$ decays as $m$ increases. However, suppose the finite-difference parameters are $h = 0.1$ and $r = 3$; then the odd–even hopscotch solution at $x = 0.8$ after eight time steps is greater than the boundary value at $x = 0$. Even though this unrealistic value will decrease rapidly from that stage onwards, due to the influence of the zero boundary condition at $x = 1$, the estimates of the temperature throughout the entire range (0, 1) are now significantly in error.

Consider further Eqs. (7) and (8) for the points $q = 2m - 1$ and $q = 2m - 2$. Again, after some algebraic manipulation, it is possible to produce a single recurrence relation in $u_{2m-1}^{2m}$. Suppressing the details, it can be shown that $u_{2m-1}^{2m}$ satisfies

$$(1 + 2r)^2\, u_{2m+1}^{2m+2} = 4r^2 u_{2m-1}^{2m} + r\left(\frac{2r}{1 + 2r}\right)^{2m}, \qquad m = 1, \ldots, n - 2. \tag{13}$$

Taking the value in (10) as the initial condition, the recurrence relation has the solution

$$u_{2m-1}^{2m} = \left[\frac{m - 1}{2(1 + 2r)} + \frac{(1 + r)^2}{(1 + 2r)}\right]\left(\frac{2r}{1 + 2r}\right)^{2m-1}, \qquad m = 1, \ldots, n - 1.$$

From the above

$$u_{2m+1}^{2m} < \tfrac{1}{2} u_{2m}^{2m}, \qquad m = 1, \ldots, n - 1, \text{ for all } r,$$

and

$$u_{2m}^{2m} > u_{2m-1}^{2m}$$

provided

$$0 < m < \min(2r^2 - 2r - 1,\, n - 1).$$

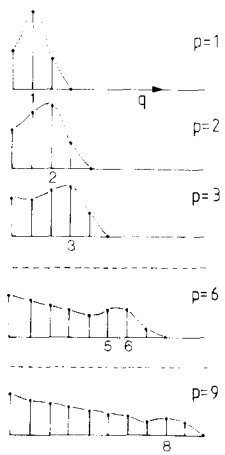


FIG. 2. Solution development for increasing $p$ ($r = 2$, $n = 5$).

Hence, for a given value of $r$ greater than about 1.4, a "wave-like" disturbance may be observed propagating through the solution domain for at least $2m$ time steps where

$$m = \min([2r^2 - 2r - 1], n - 1).$$

An example of this phenomenon is illustrated in Fig. 2.

Clearly the small time behaviour described above is inconsistent with the problem defined in (1) to (4).

## 4. Alternative Implementation

In the odd–even hoscotch approximation, Eq. (5), suppose $\theta_q^p$ is redefined as

$$\begin{aligned} \theta_q^p &= 1, \qquad p + q \text{ even}, \\ &= 0, \qquad p + q \text{ odd}. \end{aligned} \tag{14}$$

The initial approximation immediately adjacent to the discontinuity is now implicit. The propagation of non-zero values is illustrated in Fig. 3. On comparison of Figs. 1 and 3 it can be seen that the alternative hopscotch solution does not penetrate the solution domain as rapidly as that of the earlier implementation. However, the propagation of values described by (7), (8) and (9) still apply, but between odd-numbered time levels. Hence, some of the earlier results can be utilised in the examination of this alternative process. For instance, relations (11) and (12) remain valid in terms of the two leading non-zero values, $v_{2m}^{2m+1}$ and $v_{2m+1}^{2m+1}$, where $v_q^p$ represents the solution obtained by the alternative implementation, and it is readily shown that

$$v_{2m}^{2m+1} = \left(\frac{2r}{1+2r}\right)^{2m}, \qquad m = 0, 1, \ldots, n - 1.$$

Clearly $1 > v_{2m}^{2m+1} > v_{2m+1}^{2m+1}$ for all $r > 0$ and $0 < m \leqslant n - 1$. Following the same procedure as that outlined in the preceding section the behaviour of $v_{2m-1}^{2m+1}$ is obtained. Again suppressing detail, the solution for $v_{2m-1}^{2m+1}$, during the early stages of the odd–even hopscotch process, is given by

$$v_{2m-1}^{2m+1} = \frac{1}{2}\left[\frac{2m}{(1+2r)^2} + \frac{(1+8r+8r^2)}{(1+2r)^2}\right]\left(\frac{2r}{1+2r}\right)^{2m-1}, \qquad m = 1, \ldots, n - 1.$$

A little further analysis shows that

$$1 > v_{2m-1}^{2m+1} > v_{2m}^{2m+1}, \qquad \text{for all } r > 0 \text{ and } 1 < m \leqslant n - 1.$$

The leading non-zero values of the solution obtained from the alternative implementation of the odd-even hopscotch approximation do not exhibit the previously observed behaviour.

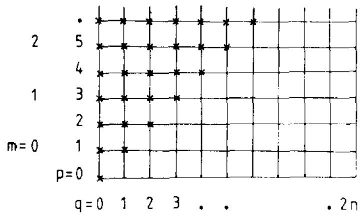


FIG. 3. Alternative propagation of non-zero values ($x$ denotes non-zero values).

Computational evidence suggests that, like the leading values, the solution as a whole is monotonic and mimics the behaviour expected of the temperature. This does not necessarily mean that the accuracy of the solution $v_q^p$ is good. The initial jump discontinuity at $x=0$ will introduce errors into both solutions. In the latter implementation the errors introduced do not create an obviously erroneous temperature distribution.

## 5. REMARKS

The above analysis illustrates the possible consequences of ignoring discontinuities when solving heat flow problems by numerical techniques. The odd–even hopscotch process seems unusually sensitive in this respect although it has been demonstrated how to overcome this sensitivity. The alternative implementation can be thought of as simply introducing a suitable starting value at the first internal mesh point and, in effect, smoothing out the jump discontinuity, which is a remedy that has been suggested in connection with other numerical techniques. It must be stressed that for problems with smooth initial data there is no significant difference between the solutions $u_q^p$ and $v_{q.}^p$ .

Finally, it is worth examining the special case $r=0.5$. A review of the solutions in the preceding two sections reveal that

$$v_{2m-1}^{2m+1} = u_{2m-1}^{2m},$$
$$v_{2m}^{2m+1} = u_{2m}^{2m},$$

and

$$v_{2m+1}^{2m+1} = u_{2m+1}^{2m}, \qquad m = 1,\dots, n-1.$$

In particular, when $m = 1$ the non-zero values in both solutions are identical but out of phase by one time step. Since the hopscotch process in the alternative implementation is the same between odd time levels as the original scheme between even ones, then

$$v_q^{p+1} \equiv u_q^p \qquad \text{for all} \quad p > 1 \text{ and } 0 \leqslant q \leqslant 2n.$$

The two solutions are the same but $v_q^p$ lags behind $u_q^p$ by one time step. A comparison of the computed values with those obtained from the analytic solution [5], at various stages in the process, suggests that $v_q^p$ is the more accurate solution.

Hence, it appears prudent to adopt the alternative implementation (definition (14) for $\theta_q^p$) as a matter of course when using the odd–even hopscotch technique.

Although the foregoing analysis has been confined to the simple, one-dimensional, heat conduction equation it may be anticipated that the wave-like behaviour demonstrated in this note could occur in a hopscotch solution of a more general heat flow problem. For example, the "line" hopscotch solution of a two-dimensional thermal shock problem will exhibit similar phenomena if not implemented with care. The results presented here illustrate the dangers involved in the casual use of hopscotch techniques.

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